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On a perturbed functional integral equation of Urysohn type

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ABSTRACT

We study the existence of monotonic solutions for a perturbed functional integral equation of Urysohn type in the space of Lebesgue integrable functions on an unbounded interval. The technique associated with measures of noncompactness (in both the weak and the strong sense) and the Darbo fixed point are the main tool to prove our main result.

1. Introduction

Nonlinear integral equations appear in many applications. For example, they occur in solving several problems arising in economics, engineering and physics. The most frequently investigated nonlinear integral equations are the Hammerstein integral equation and its generalization, the Urysohn integral equation (cf. [1–13]),

$$x(t) = g(t) + \int_I u(t,s,x(s))\,ds, \quad t \in I, \tag{1.1}$$

where I is an interval in $\mathbb{R}$ (bounded or not) and $g : I \to \mathbb{R}$, $u : I \times I \times \mathbb{R} \to \mathbb{R}$ are given functions while $x : I \to \mathbb{R}$ is an unknown function. In the case when I is a bounded interval the theory of Eq. (1.1) is well developed, and we refer to [9,10,14,15] and references therein for existence results as well as applications to other questions. On the other hand, there are few papers that consider the case when I is an unbounded interval, see [16–18].

In this paper, we study the problem of existence of monotonic solutions for the functional integral equations of Urysohn type

$$x(t) = f_1(t, x(\phi(t))) + f_2\left(t, \int_0^\infty u(t,s,x(\phi(s)))\,ds\right), \quad t \in \mathbb{R}_+ = [0,\infty). \tag{1.2}$$

Throughout $f_1, f_2 : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$, $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ are functions which satisfy special hypotheses, see Section 3. Let us recall that the function $f = f(t,x)$ involved in Eq. (1.2) generates the superposition operator F defined by

$$(Fx)(t) = f(t, x(t)), \tag{1.3}$$

where $x = x(t)$ is an arbitrary function defined on $\mathbb{R}_+$, see [19].

Eq. (1.2) has a rather general form and contains as special cases many functional integral equations, see for example [20–22]. Also, it generalizes many types of integral equations. For example, in the case $f_1(t,y) = g(t)$ and $f_2(t,v) = v$ we get an integral equation of Urysohn type studied by the author and others in [18], while in the case $f_1(t,y) = g(t)$, $f_2(t,v) = v$ and $\phi(t) = 1$ we get the famous Urysohn integral equation studied by Banaś and Pasławska-Południak in [16].

The aim of this paper is to prove the existence of monotonic solutions of Eq. (1.2) in the space of Lebesgue integrable functions on an unbounded interval. Our proof depends on a suitable combination of the Darbo fixed point theorem and

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the technique associated with both measures of weak noncompactness and measures of noncompactness in the strong sense. In fact, our results in this paper extend the technique developed by Banaś and Pasławska-Południak [16] to a more general equation such as Eq. (1.2) and generalize the result of Darwish in [17].

2. Notation and auxiliary facts

This section is devoted to collecting some definitions and results which will be needed further on.

Let $L^1(A\Omega)$ denote the space of Lebesgue integrable functions on the measurable set Ω with the standard norm

$$\|y\| = \int_\Omega |y(t)|\,dt.$$

Let us assume that $I \subset \mathbb{R}$ is a given interval, bounded or unbounded. A function $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions if it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then, to every function $x = x(t)$ which is measurable on the interval I, we may assign the function $(Fx)(t) = f(t, x(t)),\ t \in I$. The function Fx is measurable and the operator F defined in such a way is called the superposition operator generated by the function f, see [19] and references therein. The necessary and sufficient condition guaranteeing that the superposition operator F is a self-continuous map present in the following theorem. The case when I is a bounded interval was proved by Krasnosel'skii [23], while the case when I is an unbounded interval was proved by Appell and Zabrejko [19].

Theorem 1. *The superposition operator F generated by the function f maps the space $L^1(I)$ continuously into itself if and only if*

$$|f(t,x)| \leqslant a(t) + b|x|$$

for all $t \in I$ and all $x \in \mathbb{R}$, where $a \in L^1(I)$ and $b \geqslant 0$ is a constant.

Next, we recall some basic facts concerning measures of noncompactness [24,25]. Let us assume that E is an infinite dimensional Banach space with norm $\|\cdot\|$ and zero element θ. Denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E,\ \mathfrak{N}_E^{\mathcal{W}}$ its subfamilies consisting of all relatively compact and relatively weakly compact sets, respectively. For a subset X of $\mathbb{R}$, the symbols $\overline{X}$, $\overline{X}^{\mathcal{W}}$ stand for the closure and the weak closure of a set X, respectively. The symbol $\overline{\text{co}}X$ will denote the convex closed hull (with respect to the norm topology) of a set X. We denote by $B(x,r)$ the ball centered at x and of radius r. We write B_r instead of $B(\theta, r)$.

Definition 1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(1) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E^{\mathcal{W}}$.
(2) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.
(3) $\mu(\overline{X}) = \mu(\overline{\text{co}}X) = \mu(X)$.
(4) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\,\mu(Y)$ for $0 \leqslant \lambda \leqslant 1$.
(5) If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n,\ X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$ and if $\lim_{n\to\infty}\mu(X_n) = 0$ then $\cap_{n=1}^{\infty} X_n \neq \phi$.

The family $\ker\mu$ described above is called the kernel of the measure of noncompactness μ.

Definition 2. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of weak noncompactness in E if it satisfies the following conditions: (2)–(4) of Definition 1 and the following two conditions:

(1') The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E^{\mathcal{W}}$.
(5') If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n^{\mathcal{W}},\ X_{n+1} \subset X_n$ for $n = 1, 2, 3, \ldots$and if $\lim_{n\to\infty}\mu(X_n) = 0$ then $\cap_{n=1}^{\infty} X_n \neq \phi$.

Definition 3 [24]. Let X be a bounded subset of E. The Hausdorff measure of noncompactness χ is defined by

$$\chi(X) = \inf\{\varepsilon > 0 :\ \text{there is a finite subset } Y \text{ of } E \text{ suchthat } X \subset Y + B_\varepsilon\}.$$

The first important and convenient measure of noncompactness β was defined by De Blasi [26]

$$\beta(X) = \inf\{\varepsilon > 0 :\ \text{there is a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + B_\varepsilon\}.$$

The measures of noncompactness χ and β have some interesting properties. They play a significant role in nonlinear analysis and find many applications (cf. [24–27]).

We will make use of the following fixed point theorem due to Darbo [28]. To quote this theorem, we need the following definition.

Definition 4. Let M be a nonempty subset of a Banach space E and let $\mathcal{P}: M \to E$ be a continuous operator which transforms bounded sets into bounded ones. We say that $\mathcal{P}$ satisfies the Darbo condition (with constant $q \geqslant 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(\mathcal{P}X) \leqslant q\mu(X).$$

If $\mathcal{P}$ satisfies the Darbo condition with $q < 1$ then it is called a contraction operator with respect to μ.

Theorem 2 [28]. *Let Q be a nonempty, bounded, closed and convex subset of the space E and let*

$$\mathcal{P}: Q \to Q$$

be a continuous mapping which is a contraction with respect to the measure of noncompactness μ.

Then $\mathcal{P}$ has at least one fixed point in the set Q.

Remark 1 [29]. Theorem 2 remains valid if μ is a measure of weak noncompactness and if we assume that $\mathcal{P}$ is a weakly continuous map.

Also, we recall a theorem concerning the compactness in measure of a subset X of $L^1(I)$, see [30].

Theorem 3. *Let X be a bounded subset of $L^1(I)$ consisting of all functions which are a.e. nondecreasing (or nonincreasing) on the interval I. Then X is compact in measure.*

In what follows, we will work on the space $L^1(\mathbb{R}_+)$. We recall the formula for a measure of weak noncompactness, see [16,31]. Let us fix a bounded subset X of $L^1(\mathbb{R}_+)$ and define

$$c(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_\Omega |x(t)|\, dt : \Omega \subset \mathbb{R}_+, \ \mathrm{meas}(\Omega) \leqslant \varepsilon \right] \right\} \right\}$$

and

$$d(X) = \lim_{T \to \infty} \left\{ \sup \left[\int_T^\infty |x(t)|\, dt : x \in X \right] \right\}.$$

Put

$$\gamma(X) = c(X) + d(X). \tag{2.4}$$

Then we have the following results, see [16,31] and references therein.

Theorem 4. *The function γ is a measure of weak noncompactness in the space $L^1(\mathbb{R}_+)$ such that*

$$\beta(X) \leqslant \gamma(X) \leqslant 2\beta(X),$$

where β denotes the De Blasi measure of noncompactness. Moreover, $\gamma(B_{L^1(\mathbb{R}_+)}) = 2$.

Theorem 5. *Let X be a nonempty, bounded and compact in measure subset of the space $L^1(\mathbb{R}_+)$. Then*

$$\chi(X) \leqslant \gamma(X) \leqslant 2\chi(X).$$

Theorem 6. *Let X be a nonempty, bounded and compact in measure subset of the space $L^1(\mathbb{R}_+)$. If $\mathcal{P}: X \to L^1(\mathbb{R}_+)$ is a continuous map then it is weakly sequentially continuous on X.*

By combining all the above established facts (Theorems 3, 5, 6 and 2) we can deduce the following result [16].

Theorem 7. *Let Q be a nonempty, bounded, closed and convex and compact in measure subset of the space $L^1(\mathbb{R}_+)$ and let*

$$\mathcal{P}: Q \to Q$$

be a continuous mapping which is a contraction with respect to the measure of weak noncompactness γ.

Then $\mathcal{P}$ has at least one fixed point in the set Q.

3. Main result

In this section, we will study Eq. (1.2) assuming that the following hypotheses are satisfied:

(h_1) The function $f_i(t,x) = f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfies the Carathéodory conditions, and there exist a function $a_i \in L^1(\mathbb{R}_+)$ and a constant $b_i > 0$ such that

$$|f_i(t,x)| \leqslant a_i(t) + b_i|x|, \quad i = 1,2,$$

for $t \in \mathbb{R}_+$ and for $x \in \mathbb{R}$. Moreover, $f_i(t,x),\ i = 1,2$, are assumed to be nonincreasing with respect to t and nondecreasing with respect to x.

(h_2) The function $u(t,s,x) = u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfies the Carathéodory conditions, and there exist a function $a_3 \in L^1(\mathbb{R}_+)$ and a constant $b_3 > 0$ such that

$$|u(t,s,x)| \leqslant k(t,s)[a_3(t) + b_3|x|]$$

for $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$ and for $x \in \mathbb{R}$, where the function $k(t,s) = k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is measurable such that the linear Fredholm integral operator

$$(Kx)(t) = \int_0^\infty k(t,s)x(s)\,ds$$

transforms the space $L^1(\mathbb{R}_+)$ into itself and is continuous.

(h_3) The function $t \to u(t,s,x)$ is a.e. nonincreasing on $\mathbb{R}_+$ for almost all $s \in \mathbb{R}_+$ and for each $x \in \mathbb{R}$.

(h_4) The function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and absolutely continuous. Moreover, there is a constant $M > 0$ such that $\phi'(t) \geqslant M$ for almost all $t \geqslant 0$.

(h_5) $b_1 + b_2 b_3\|K\| < M$.

Remark 2. In (h_2), when we say that the function $u(t,s,x) = u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ satisfies the Carathéodory conditions, we mean that the function $(t,s) \to u(t,s,x)$ is measurable for any $x \in \mathbb{R}$ and the function $x \to u(t,s,x)$ is continuous for almost all $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

For further purposes let us denote by U the Urysohn integral operator generated by the function u, i.e.,

$$(Ux)(t) = \int_0^\infty u(t,s,x(s))\,ds. \tag{3.5}$$

Thus Eq. (1.2) takes the form

$$x = \mathcal{F}x = F_1x(\phi) + F_2Ux(\phi), \tag{3.6}$$

where F_1 and F_2 are the superposition operators generated by the functions $f_1(t,x)$ and $f_2(t,x)$, respectively.

Remark 3 ([13,16]). Under the hypothesis (h_2) the Urysohn integral operator (3.5) maps $L^1(\mathbb{R}_+)$ continuously into itself.

Now, we are in a position to state and prove our main result.

Theorem 8. *Let the hypotheses (h_1)–(h_5) be satisfied. Then Eq. (1.2) has at least one solution $x \in L^1(\mathbb{R}_+)$ which is a.e. nonincreasing on $\mathbb{R}_+$.*

Proof. First, observe that for a given $x \in L^1(\mathbb{R}_+)$, we have that $\mathcal{F}x \in L^1(\mathbb{R}_+)$ and also $\mathcal{F}$ is continuous in $L^1(\mathbb{R}_+)$, thanks to our hypotheses, Remark 3 and Theorem 1. Moreover, by virtue of our hypotheses we get

$$\begin{aligned}
\|\mathcal{F}x\| &\leqslant \|F_1x(\phi)\| + \|F_2Ux(\phi)\| \\
&= \int_0^\infty |f_1(t,x(\phi(t)))|\,dt + \int_0^\infty \left|f_2\left(t, \int_0^\infty u(t,s,x(\phi(s)))ds\right)\right| dt \\
&\leqslant \int_0^\infty [a_1(t) + b_1|x(\phi(t))|]\,dt + \int_0^\infty \left[a_2(t) + b_2\left|\int_0^\infty u(t,s,x(\phi(s)))\,ds\right|\right] dt \\
&\leqslant \|a_1\| + b_1\|x(\phi)\| + \|a_2\| + b_2\int_0^\infty \int_0^\infty k(t,s)a_3(s)\,ds\,dt + b_2b_3\int_0^\infty \int_0^\infty k(t,s)|x(\phi(s))|\,ds\,dt \\
&= \|a_1\| + b_1\|x(\phi)\| + \|a_2\| + b_2\|Ka_3\| + b_2b_3\|Kx(\phi)\| \\
&\leqslant \|a_1\| + b_1\|x(\phi)\| + \|a_2\| + b_2\|K\|\|a_3\| + b_2b_3\|K\|\|x(\phi)\| \\
&= \|a_1\| + \|a_2\| + b_2\|K\|\|a_3\| + (b_1 + b_2b_3\|K\|)\int_0^\infty |x(\phi(t))|\,dt \\
&\leqslant \|a_1\| + \|a_2\| + b_2\|K\|\|a_3\| + (b_1 + b_2b_3\|K\|)M^{-1}\int_0^\infty |x(\phi(t))|\phi'(t)\,dt \\
&\leqslant \|a_1\| + \|a_2\| + b_2\|K\|\|a_3\| + (b_1 + b_2b_3\|K\|)M^{-1}\|x\|,
\end{aligned}$$

where $\|K\|$ denotes the norm of the linear Fredholm integral operator mapping $L^1(\mathbb{R}_+)$ into itself. Apart from this the norm used in the above estimates denotes the norm in $L^1(\mathbb{R}_+)$.

From the last estimate we deduce that the operator $\mathcal{F}$ transforms the ball B_r into itself for $r = (\|a_1\| + \|a_2\| + b_2\|K\|\|a_3\|)/[1 - (b_1 + b_2b_3\|K\|)M^{-1}]$, thanks to hypothesis (h_5).

In what follows let us define D to be the subset of the ball B_r consisting of all functions which are a.e. positive and nonincreasing on $\mathbb{R}_+$. Then the set D is nonempty, bounded, closed and convex, see [30]. Moreover, D is compact in measure, thanks to Theorem 3.

It is easy to see that the operator $\mathcal{F}$ maps the set D into itself, thanks to hypotheses (h_1), (h_3) and (h_4).

We now show that the operator $\mathcal{F}$ is a contraction with respect to the measure of weak compactness γ. For this purpose take a nonempty subset X of D and fix $\varepsilon > 0$. Further, let us take a nonempty subset Ω of $\mathbb{R}_+$ such that Ω is measurable and $\text{meas}(\Omega) \leqslant \varepsilon$. Then for an arbitrary $x \in X$ and in view of our hypotheses we have

$$\begin{aligned}
\int_\Omega |(\mathcal{F}x)(t)|\,dt &\leqslant \int_\Omega [a_1(t) + b_1|x(\phi(t))|]\,dt + \int_\Omega \left[a_2(t) + b_2\left|\int_0^\infty u(t,s,x(\phi(s)))\,ds\right|\right]dt \\
&\leqslant \int_\Omega a_1(t)\,dt + b_1\|x(\phi)\|_{L^1(\Omega)} + \int_\Omega a_2(t)\,dt + b_2\int_\Omega\int_0^\infty k(t,s)a_3(s)\,ds\,dt + b_2b_3\int_\Omega\int_0^\infty k(t,s)|x(\phi(s))|\,ds\,dt \\
&= \int_\Omega a_1(t)\,dt + b_1\|x(\phi)\|_{L^1(\Omega)} + \int_\Omega a_2(t)\,dt + b_2\|Ka_3\|_{L^1(\Omega)} + b_2b_3\|Kx(\phi)\|_{L^1(\Omega)} \\
&\leqslant \int_\Omega a_1(t)\,dt + \int_\Omega a_2(t)\,dt + b_2\|K\|_\Omega\int_\Omega a_3(t)\,dt + (b_1 + b_2b_3\|K\|_\Omega)\|x(\phi)\|_{L^1(\Omega)}
\end{aligned}$$

where $\|K\|_\Omega$ denotes the norm of the linear Fredholm integral operator mapping $L^1(\Omega)$ into itself.

Hence, we have

$$\begin{aligned}
\int_\Omega |(\mathcal{F}x)(t)|\,dt &\leqslant \int_\Omega a_1(t)\,dt + \int_\Omega a_2(t)\,dt + b_2\|K\|\int_\Omega a_3(t)\,dt(b_1 + b_2b_3\|K\|)\int_\Omega |x(\phi(t))|\,dt \\
&\leqslant \int_\Omega a_1(t)\,dt + \int_\Omega a_2(t)\,dt + b_2\|K\|\int_\Omega a_3(t)\,dt + (b_1 + b_2b_3\|K\|)M^{-1}\int_\Omega |x(\phi(t))|\phi'(t)\,dt \\
&= \int_\Omega a_1(t)\,dt + \int_\Omega a_2(t)\,dt + b_2\|K\|\int_\Omega a_3(t)\,dt + (b_1 + b_2b_3\|K\|)M^{-1}\int_{\phi(\Omega)} |x(v)|\,dv,
\end{aligned}$$

Now, taking into account the fact that

$$\lim_{\varepsilon\to 0}\left\{\sup\left[\int_\Omega |a_i(t)|\,dt : \Omega \subset \mathbb{R}_+, \text{meas}(\Omega) \leqslant \varepsilon\right]\right\} = 0, \quad i = 1,2,3$$

and keeping in mind that the function ϕ is assumed to be absolutely continuous, from the last estimate we have

$$c(\mathcal{F}X) \leqslant qc(X), \tag{3.7}$$

where $q = (b_1 + b_2b_3\|K\|)M^{-1}$. Obviously, in view of hypothesis (h_5) we have that $q < 1$.

Next, let us fix an arbitrary number $T > 0$. Then, taking into account our hypotheses, for an arbitrary function $x \in X$ we have

$$\begin{aligned}
\int_T^\infty |(\mathcal{F}x)(t)|\,dt &\leqslant \int_T^\infty a_1(t)\,dt + b_1\int_T^\infty |x(\phi(t))|\,dt + \int_T^\infty a_2(t)\,dt + b_2\int_T^\infty\left|\int_0^t u(t,s,x(\phi(s)))\,ds\right|dt \\
&\leqslant \int_T^\infty a_1(t)\,dt + b_1\int_T^\infty |x(\phi(t))|\,dt + \int_T^\infty a_2(t)\,dt + b_2\int_T^\infty\int_0^\infty k(t,s)a_3(s)\,ds\,dt + b_2b_3\int_T^\infty\int_0^t k(t,s)|x(\phi(s))|\,ds\,dt \\
&= \int_T^\infty a_1(t)\,dt + b_1\int_T^\infty |x(\phi(t))|\,dt + \int_T^\infty a_2(t)\,dt + b_2\|Ka_3\|_{L^1([T,\infty))} + b_2b_3\|Kx(\phi)\|_{L^1([T,\infty))} \\
&\leqslant \int_T^\infty a_1(t)\,dt + \int_T^\infty a_2(t)\,dt + b_2\|K\|_{[T,\infty)}\int_T^\infty a_3(t)\,dt + (b_1 + b_2b_3\|K\|_{[T,\infty)})\int_T^\infty |x(\phi(t))|\,dt \\
&\leqslant \int_T^\infty a_1(t)\,dt + \int_T^\infty a_2(t)\,dt + b_2\|K\|\int_T^\infty a_3(t)\,dt + (b_1 + b_2b_3\|K\|)M^{-1}\int_T^\infty |x(\phi(t))|\phi'(t)\,dt \\
&\leqslant \int_T^\infty a_1(t)\,dt + \int_T^\infty a_2(t)\,dt + b_2\|K\|\int_T^\infty a_3(t)\,dt + (b_1 + b_2b_3\|K\|)M^{-1}\int_{\phi(T)}^\infty |x(v)|\,dv,
\end{aligned}$$

where $\|K\|_{[T,\infty)}$ denotes the norm of the linear Fredholm integral operator mapping $L^1([T,\infty))$ into itself. Hence, from the last estimate we obtain

$$d(\mathcal{F}X) \leqslant qd(X), \tag{3.8}$$

due to the fact that $d(Y) = 0$ for any singleton Y of $L^1(\mathbb{R}_+)$. From (3.7) and (3.8) and the definition of the measure of noncompactness γ given by formula (2.4), we obtain

$$\gamma(\mathcal{F}X) \leqslant q\gamma(X). \tag{3.9}$$

Now, an application of Theorem 7 with $Q = D$ and $\mathcal{P} = \mathcal{F}$ implies that the operator $\mathcal{F}$ has at least one fixed point in D. This completes the proof. □

4. An example

Consider the functional integral equation of Urysohn type

$$x(t) = 2e^{-t^2} + \frac{1}{8}\arctan x^2(t) + \arctan\left(\int_0^\infty (t+s)e^{-t}\left[\frac{s}{s^2+1} + \frac{1}{8}\ln(1+x^2(s))\right]ds\right)^2. \quad (4.10)$$

In this example, we have that $f_1(t,x) = e^{-t^2} + \frac{1}{8}\arctan x^2$ and $f_2(t,x) = e^{-t^2} + \arctan x^2$ and this function satisfies hypothesis (h_1) with $a_i(t) = e^{-t^2},\ (i = 1,\ 2), b_1 = \frac{1}{4}$ and $b_2 = 2$, because of the fact that $\arctan x^2 \leqslant 2x$ for $x \geqslant 0$. Moreover,

$$u(t,s,x) = (t+s)e^{-t}\left[\frac{s}{s^2+1} + \frac{1}{8}\ln(1+x^2(s))\right]$$

satisfies hypotheses (h_2) and (h_3) with $k(t,s) = (t+s)e^{-t},\ a_3(t) = \frac{t}{t^2+1}$ and $b_3 = \frac{1}{8}$, since $\ln(1+x^2) \leqslant x$. Also, $\|K\| = \frac{2}{\sqrt{e}}$, see [32]. We have $\phi(t) = t$ and so $\phi'(t) = 1$. Thus, $M = 3/4$,

$$b_1 + b_2 b_3\|K\| = \frac{1}{4} + \frac{1}{2\sqrt{e}} \leqslant M.$$

Therefore, Theorem 8 guarantees that Eq. (4.10) has a solution $x = x(t)$ in the space $L^1(\mathbb{R}_+)$ which is a.e. nonincreasing on $\mathbb{R}_+$.

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